New phenomena in the random field Ising model

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Abstract

We reconsider Ising spins in a Gaussian random field within the replica formalism. The corresponding continuum model involves several coupling constants beyond the single one which was considered in the standard ϕ^4 theory approach. These terms involve more than one replica, and therefore in a mean field theory they do not contribute to the zero-replica limit. However the fluctuations involving those extra terms are singular on the Curie line below eight dimensions, and by the time one reaches the dimension six, it is necessary to keep them in the renormalization group analysis. As a result it is found that there is no stable fixed point of order $(6 - d)$. Whether this means that there is no expansion in powers of $(6 - d)$, or that the transition is driven to first order by these fluctuations, is difficult to decide at this level, but it explains the failure of the $(d, d - 2)$ correspondence.

In spite of more than twenty years of active and continuous research, the simplest disordered system, the random field Ising model (RFIM), is still far from a definite understanding. The initial (d, d-2) correspondence, between the RFIM in dimension d and the pure system in dimension (d-2)([1, 2, 3]), was suspected from the very beginning to break down in lower dimensions, if one followed the arguments of Imry and Ma ([4]). The correspondence was definitely killed when the mathematical approach of Imbrie, and subsequently Bricmont and Kupiainen, finally established the existence of an ordered state at low temperature in three dimensions([5, 6]). But this still left room for various possibilities. The elegant "derivation" of the (d, d-2) relation by Parisi and Sourlas ([3]) made it clear that non-perturbative issues could destroy the essential underlying supersymmmetry (since a Jacobian, the absolute value of a determinant, was replaced by the determinant itself which could change sign). One could thus imagine that the (d, d-2) correspondence for the critical exponents, although true to all orders in an expansion in powers of $\epsilon = 6 - d$, was modified by terms, such as $\exp -1/\epsilon$, which might very well turn out to be quite significant at $\epsilon = 3$. However progressive evidence was provided that the situation could be more serious than that. Numerical experiments in the supposedly paramagnetic phase, pointing at large time scales before one reached the ordering temperature, as well as the theoretical approach of Mézard and Young ([7]), led to the idea that the phase diagram itself had to be reconsidered. Coming from the high temperature regime it seems that one meets a glassy phase, marked by replica symmetry breaking (RSB), in a region in which there is no magnetic ordering, as in a spin glass. We shall present in a subsequent publication arguments which substantiate this analysis. However here we argue that within the original models of Ising spins, or of the ϕ^4 theory, in a Gaussian random field, another scenario might take place. Indeed we show, within the replica approach, that new interaction terms should be considered. These terms naively disappear in the zero replica limit, but it is argued that below dimension eight, these new interactions develop singularities near the Curie line in the $n = 0$ limit, which change the analysis when one reaches the dimension six. The critical renormalization group flow, if performed as usual on the Curie line, contains then five coupling constants, and it is easy to show that there is no stable fixed point of order (6-d). Above the Curie line these singularities are cut-off by the finiteness of the correlation length, and there is a non-uniformity in the approach of the Curie line, versus the zero-replica

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limit. We are not able to say whether the absence of a stable fixed-point means that there is a non-perturbative fixed point elsewhere or whether the transition becomes first order, but it shows that the dimensional reduction may break down even at lowest order.

1   The basic action

We consider successively, within the replica formalism, ϕ^4 and Ising spins in an external Gaussian random field $h(x)$ with

$$< h(x) >= 0; < h(x)h(y) >= \Delta\delta(x-y). \tag{1.1}$$

Therefore starting with

$$S(\phi) = \int dx[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}r_0\phi^2 + \frac{g}{8}\phi^4 - h(x)\phi(x)] \tag{1.2}$$

(the Boltzmann weight is $exp - S$) and replicating n times we average over the random field and obtain

$$S(\phi_\alpha) = \int dx[\sum_{\alpha=1}^{n}(\frac{1}{2}(\nabla\phi_\alpha)^2 + \frac{1}{2}r_0\phi_\alpha^2 + \frac{g}{8}\phi_\alpha^4) - \frac{\Delta}{2}\sum_{\alpha,\beta}\phi_\alpha(x)\phi_\beta(x)] \tag{1.3}$$

Let us introduce the notation

$$\sigma_k = \sum_{\alpha=1}^{n}(\phi_\alpha(x))^k; \tag{1.4}$$

the symmetry of this action under permutation of the replicas allows for five, a priori relevant, coupling constants corresponding to the quartic interactions $\sigma_4, \sigma_1\sigma_3, (\sigma_2)^2, (\sigma_1)^2\sigma_2, (\sigma_1)^4$.

Indeed if we were considering the renormalization programme for the various correlation functions of the theory at finite n , we would be forced to introduce these coupling constants, even if they are not all present initially. In order to substantiate this claim, let us start with Ising spins and derive the corresponding ϕ^4 theory in the usual way. We begin with

$$\beta H = -\frac{1}{2}\sum_{i,j}J_{ij}S_iS_j - \sum_i h_iS_i, \tag{1.5}$$

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and use the Hubbard-Stratonovich identity

$$\exp[\frac{1}{2}\sum J_{ij}S_iS_j+\sum h_iS_i] = C\int(\prod_i d\phi_i)\exp[-\frac{1}{2}\sum \phi_i J_{ij}^{-1}\phi_j+\sum(\phi_i+h_i)S_i]$$
(1.6)

We then introduce n replicas and sum over the spins. This gives the partition function

$$Z^n = C\int \prod_{i=1}^{N}\prod_{\alpha=1}^{n} d\phi_{i,\alpha}\exp[-\frac{1}{2}\sum_{ij,\alpha}\phi_{i,\alpha}J_{ij}^{-1}\phi_{j,\alpha} + \sum_{i,\alpha}\ln\cosh(\phi_{i,\alpha}+h_i)]$$ (1.7)

We then expand in powers of ϕ, up to degree four:

$$\ln\frac{\cosh(\phi+h)}{\cosh h} = \ln(1+\tau\phi+\frac{1}{2}\phi^2+\frac{\tau}{6}\phi^3+\frac{1}{24}\phi^4+O(\phi^5))$$ (1.8)

in which $\tau = \tanh h$, and average over the random field h:

$$< \exp\sum_\alpha \ln\frac{\cosh(\phi_\alpha+h)}{\cosh h} >= \exp(\frac{1}{2}(1-\tau_2)\sigma_2 + \frac{\tau_2}{2}\sigma_1^2$$

$$-\frac{1}{12}(1-4\tau_2+3\tau_4)\sigma_4 + \frac{1}{24}(\tau_4-3\tau_2^2)\sigma_1^4 + \frac{1}{8}(\tau_4-\tau_2^2)\sigma_2^2$$

$$+\frac{1}{3}(\tau_4-\tau_2)\sigma_1\sigma_3 - \frac{1}{4}(\tau_4-\tau_2^2)\sigma_1^2\sigma_2 + O(\phi^6))$$ (1.9)

with
$$\tau_p = \frac{1}{\sqrt{2\pi\Delta}}\int_{-\infty}^{+\infty}dh e^{-\frac{h^2}{2\Delta}}(\cosh h)^n(\tanh h)^p$$ (1.10)

At the level of the quadratic terms $(1-\tau_2)\sigma_2$ shifts the constant interaction term $\tilde{J}^{-1}(q=0)$ and gives an effective 'mass'. The other term $\tau_2\sum\phi_\alpha\phi_\beta$ is due to the average over the random field. For a small random field, i.e. small Δ, in the $n=0$ limit

$$\tau_2 = \Delta - 2\Delta^2 + O(\Delta^3); \tau_4 = 3\Delta^2 - 20\Delta^3 + O(\Delta^4)$$ (1.11)

We are then led to a ϕ^4- theory with five coupling constants:

$$\beta H = \int d^dx(\frac{1}{2}\sum_\alpha[(\nabla\phi_\alpha)^2 + t\phi_\alpha^2] - \frac{\Delta}{2}\sum_{\alpha,\beta}\phi_\alpha\phi_\beta$$

$$+\frac{u_1}{4!}\sigma_4 + \frac{u_2}{3!}\sigma_1\sigma_3 + \frac{u_3}{8}\sigma_2^2 + \frac{u_4}{4}\sigma_1^2\sigma_2 + \frac{u_5}{4!}\sigma_1^4)$$ (1.12)

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A priori the four additional terms beyond the single interaction σ_4, which is the only one usually present, involve more than one single replica. Therefore if one writes the quenched average of the replicated partition function in terms of $W = \mathrm{Log} Z$,

$$< Z^n >= \exp[n < W > + \frac{n^2}{2}(< W^2 > - < W >^2) + O(n^3)], \qquad (1.13)$$

it is easily seen that the contribution of the four additional coupling constants are of relative order n (or higher) and thus could be discarded. However this argument is true provided the coupling constants u_i for $i > 1$, are not singular in the $n = 0$ limit.

2 Singularities below dimension eight

At the level of mean field theory the coupling constants that we have obtained in (1.9) are all finite in the $n = 0$ limit. However the situation changes with the fluctuations: consider the one-loop fluctuations in the theory at the critical point. These fluctuations are given by the determinant generated by the Gaussian fluctuations around the mean field, namely

$$\exp -\frac{1}{2}\mathrm{Tr}\ln(1 + \mathrm{G_0 S}^{(2)}) \qquad (2.1)$$

in which

$$G_0^{\alpha\beta}(q) = \frac{\delta_{\alpha\beta}}{(q^2 + t)} + \frac{\Delta}{(q^2 + t)(q^2 + t - n\Delta)} \qquad (2.2)$$

and $S^{(2)}$ is the second derivative of the action (1.12) with respect to ϕ_α and ϕ_β

$$S_{\alpha\beta}^{(2)} = \delta_{\alpha\beta}[\frac{1}{2}u_1\phi_\alpha^2 + u_2\sigma_1\phi_\alpha + \frac{1}{2}u_3\sigma_2 + \frac{1}{2}u_4\sigma_1{}^2] + \frac{1}{2}u_2(\phi_\alpha^2 + \phi_\beta^2)$$
$$+ u_3\phi_\alpha\phi_\beta + \frac{1}{2}u_4\sigma_2 + u_4\sigma_1(\phi_\alpha + \phi_\beta) + \frac{1}{2}u_5\sigma_1{}^2 \qquad (2.3)$$

in which we have used the notation (1.4). The fluctuations terms which are quartic in ϕ are obtained by expanding (2.1) to second order in $S^{(2)}$. We first

consider the most singular part in which for the propagator G_0 we retain twice the Δ part of (2.2). It involves the sum

$$\Delta^2(\sum_{\alpha\beta} S^{(2)}_{\alpha\beta})^2 = \Delta^2[\frac{1}{2}(u_1 + 4nu_2 + nu_3 + n^2u_4)\sigma_2$$

$$+(u_2 + u_3 + \frac{5}{2}nu_4 + n^2u_5)\sigma_1^2]^2 \qquad (2.4)$$

multiplied by the one-loop integral

$$I_n(p, t) = \int d^dq \frac{1}{(q^2 + t)(q^2 + t - n\Delta)((p - q)^2 + t)((p - q)^2 + t - n\Delta)}. \qquad (2.5)$$

The contribution of these fluctuations to the effective coupling constants are thus related to $I_n(0, t)$. Note that one cannot let t go to zero first, since this would yield a pole of the integrand on the line $q^2 = n\Delta$. Therefore in order to determine the critical behaviour of this effective interaction we have to examine $I_n(0, t)$ for t of order $n\Delta$, and then let n go to zero. On the line $t = n\Delta$, which is close to the usual critical $t = 0$ line, the integral $I_n(0, t)$ becomes singular, in dimensions lower than eight. A simple power counting gives

$$I_n(0, n\Delta) = \int d^dq \frac{1}{q^4(q^2 + n\Delta)^2} \sim (n\Delta)^{-\frac{8-d}{2}}. \qquad (2.6)$$

Therefore these fluctuations yield singular contribution to the coupling constants contained in (2.4) in dimensions lower than eight in the zero-replica limit. Thus the coupling constant u_3 receives a contribution proportional to $\frac{1}{n}u_1^2$ in dimension six, likewise for u_2. On the other hand u_4 behaves like u_1u_3/n, i.e. like $1/n^2$ and u_5 like u_3^2/n, i.e. $1/n^3$. Consequently it is not legitimate to discard the coupling constants which couple several replicas in dimensions smaller than eight.

It is interesting to analyze how a dynamical approach, based for instance on a Langevin formalism, would recover those singularities, despite the absence of replica. In the Martin-Rose-Siggia formulation of the dynamics ([8]), there are two fields, namely $\phi(x, t)$ the order parameter, and the conjugate field $\hat{\phi}(x, t)$. The equilibrium ϕ^4 theory leads to a single dynamical interaction term of the form $\hat{\phi}\phi^3$. However fluctuations build up additional couplings such as $\phi^2\hat{\phi}^2$,etc., with singular coefficients in dimensions lower than eight.

It is then necessary to introduce a waiting time and invariance under time-translation invariance is broken . The $n = 0$ singularity in the critical domain is then replaced by a singularity when the waiting time becomes large. A full comparison between the dynamical approach and the equilibrium replica formalism goes beyond the scope of this letter.

3 The renormalization group flow near six dimensions

Since the multi-replica coupling constants are singular below dimension eight, one cannot discard the new interaction terms in the zero-replica limit, since they blow up as powers of $1/n$ in dimension six. Therefore we keep now the full action (1.12) with the five coupling constants, and repeat the usual renormalization group approach near six dimensions. We know that if we had kept only the coupling constant u_1 we would have found a stable fixed point of order $\epsilon = 6 - d$, which would yield a critical behaviour identical to that of the pure system near dimension four([2, 3]). At one-loop we obtain the renormalization group equations by returning to (2.1), expanding again to second order in $S^{(2)}$, but now keeping only one Δ in the propagators. The result is proportional to

$$
\begin{aligned}
\sum_{\alpha}(\sum_{\beta} S_{\alpha\beta}^{(2)})^2 = \quad & \frac{1}{4}(u_1 + nu_2)^2\sigma_4 + (u_1 + nu_2)(u_2 + u_3 + nu_4)\sigma_1\sigma_3 + \\
+ \quad & \frac{1}{4}[n(u_2 + u_3 + nu_4)^2 + 2(u_2 + u_3 + nu_4)(u_1 + nu_2)]\sigma_2{}^2 \\
+ \quad & [2(u_2 + u_3 + nu_4)^2 + \frac{1}{2}(3u_4 + nu_5)(u_1 + 2nu_2 + nu_3 + n^2u_4)]\sigma_1{}^2\sigma_2 \\
+ \quad & [(u_2 + u_3 + nu_4)(3u_4 + nu_5) + \frac{n}{4}(u_4 + nu_5)^2]\sigma_1^4
\end{aligned}
$$

(3.1)

multiplied by an integral proportional to $1/\epsilon$. Redefining the coupling constants to contain geometric factor (surface of the unit sphere divided by $(2\pi)^d$) we obtain the five one-loop beta functions :

$$
\begin{aligned}
\beta_1 &= -\epsilon u_1 + 3\Delta(u_1 + nu_2)^2 \\
\beta_2 &= -\epsilon u_2 + 3\Delta(u_1 + nu_2)(u_2 + u_3 + nu_4)
\end{aligned}
$$

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$$\begin{aligned}
\beta_3 &= -\epsilon u_3 + \Delta[n(u_2 + u_3 + nu_4)^2 + 2(u_2 + u_3 + nu_4)(u_1 + nu_2)] \\
\beta_4 &= -\epsilon u_4 + \Delta[4(u_2 + u_3 + nu_4)^2 \\
&\quad + n(u_2 + u_3 + nu_4)(3u_4 + nu_5) + (3u_4 + nu_5)(u_1 + nu_2)] \\
\beta_5 &= -\epsilon u_5 + 12\Delta[(u_2 + u_3 + nu_4)(3u_4 + nu_5) + \frac{n}{4}(3u_4 + nu_5)^2] \quad (3.2)
\end{aligned}$$

These equations make it clear that the renormalization mixes the five operators that we are considering. If, notwithstanding the potential singularities of $u_2, \cdots, u_5$, one lets n go to zero first in the equations 3.2, one recovers the usual dimensional reduction fixed point $u_1 = \frac{\epsilon}{3\Delta} + O(\epsilon^2), u_2 = \cdots = u_5 = 0$. It is immediate to check that this fixed point is unstable and that the $u_i's$ for $i > 2$ grow. Furthermore we have seen in the previous section that the effective interactions with the symmetry of those extra coupling constants, are indeed singular when n goes to zero. Therefore instead of letting n go to zero first, we give to these coupling constants the dependence in n inherited from these singularities, i.e. we combine the n-dependence from these singularities with the sum over replica indices. Examining those beta functions 3.2, it is immediate to verify that if we redefine the coupling constants

$$g_1 = \Delta u_1, g_2 = n\Delta u_2, g_3 = n\Delta u_3, g_4 = n^2\Delta u_4, g_5 = n^3\Delta u_5 \quad (3.3)$$

the number of replicas n drops completely from the equations. Note that the dependence in n of these coupling constants is exactly consistent with the analysis of the effect in dimension six of the multi-replica singularities analyzed in the previous section. This leads to the important conclusion that the existence and stability of the fixed points is now independent of the number of replicas. In terms of the couplings g_i we now have

$$\begin{aligned}
\beta_1 &= -\epsilon g_1 + 3(g_1 + g_2)^2 \\
\beta_2 &= -\epsilon g_2 + 3(g_1 + g_2)(g_2 + g_3 + g_4) \\
\beta_3 &= -\epsilon g_3 + (g_2 + g_3 + g_4)^2 + 2(g_2 + g_3 + g_4)(g_1 + g_2) \\
\beta_4 &= -\epsilon g_4 + 4(g_2 + g_3 + g_4)^2 + (g_2 + g_3 + g_4)(3g_4 + g_5) + (3g_4 + g_5)(g_1 + g_2) \\
\beta_5 &= -\epsilon g_5 + 12(g_2 + g_3 + g_4)(3g_4 + g_5) + 3(3g_4 + g_5)^2 \quad (3.4)
\end{aligned}$$

(we have kept the same name to the beta functions, in spite of an obvious refinition by a multiplicative factor corresponding to the redefinition (3.3)).

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We now look for fixed points g_i* which are zeroes of those equations and satisfy the stability condition given by the non-negativity of the eigenvalues (or rather the real part of the eigenvalues if they were complex) of the matrix

$$\Omega_{ij} = \frac{\partial \beta_i}{\partial g_j}\Big|_{g*} \tag{3.5}$$

at the fixed point. Note that if g_1 was the only coupling, the fixed point $g_1* = \frac{\epsilon}{3}$ would be stable, but it is no longer true when the others are present since $\frac{\partial \beta_3}{\partial g_3} = -\frac{\epsilon}{3}$ being negative, the matrix Ω cannot be non-negative. The discussion of the fixed point solutions is easier in terms of the variables

$$g_1 + g_2 = x, g_2 + g_3 + g_4 = y, 3g_4 + g_5 = z. \tag{3.6}$$

and the first equation (3.4) leaves us with two possibilities a) $x = 0$ and then $\frac{\partial \beta_1}{\partial g_1}$ is negative, b) $x \neq 0$, then $x + y = \frac{\epsilon}{3}$ and $\frac{\partial \beta_3}{\partial g_3} = -\epsilon + 2(x + y)$ is negative. We then conclude again that there is no stable fixed point. Therefore, whether we take into account the singular dependence in n of the effective interactions or not, one does not find a stable fixed point of order $(6 - d)$. This still leaves us with two possibilities (i) a stable fixed point which is not of order ϵ (ii) a runaway flow indicative of a first order transition. However in both cases there is no epsilon-expansion of the critical properties of the random field Ising model, and the dimensional reduction breaks down at first order in ϵ.

4 Branched polymers

One may wonder at this stage whether the previous analysis would not endanger as well the beautiful results of Parisi-Sourlas on dimensional reduction for branched polymers([9]), for which there are only reasons to believe that it works. We shall not repeat here their original derivation but simply take the replicated field theory version of the problem, which is now a ϕ^3 theory. The action is similar to (1.12)

$$\beta H = \int d^d x \Big(\frac{1}{2} \sum_\alpha [(\nabla \phi_\alpha)^2 + t\phi_\alpha^2] - \frac{\Delta}{2} \sum_{\alpha,\beta} \phi_\alpha \phi_\beta + \frac{u_1}{3!}\sigma_3 + \frac{u_2}{2}\sigma_1\sigma_2 + \frac{u_3}{3!}\sigma_1^3 \Big). \tag{4.1}$$

Again the theory has been based on the single coupling constant u_1 since the other two drop from the problem in the zero-replica limit. The upper critical

dimension is then eight, and the authors of ([9]) proved that the epsilon-expansion about eight dimensions (based on a purely imaginary coupling constant) is identical to the $(6 - d)$ expansion for the ϕ^3 theory.

Similarly here the multi-replica coupling constants develop singularities in the zero-replica limit below dimension twelve which make it necessary to reconsider the analysis of the $(8 - d)$-expansion. Following the same lines we have obtained the three beta functions. Again n drops from the determination fo the fixed points if we redefine

$$g_1 = \Delta u_1, g_2 = n\Delta u_2, g_3 = n^2 \Delta u_3 \qquad (4.2)$$

and we find

$$
\begin{aligned}
\beta_1 &= -\frac{\epsilon}{2}g_1 - 2g_1(g_1 + g_2)^2 - \frac{1}{4}g_1(g_1 + 3g_2 + g_3)^2 \\
\beta_2 &= -\frac{\epsilon}{2}g_2 - \frac{1}{4}g_2(g_1 + 3g_2 + g_3)^2 \\
&\quad -2g_1(g_1 + g_2)(2g_2 + g_3) - 2g_2(g_1 + g_2)(g_1 + 3g_2 + g_3) \\
\beta_3 &= -\frac{\epsilon}{2}g_3 + g_3(g_1 + g_2)^2 - 3(2g_2 + g_3)^2(g_1 + g_2) - 6g_2(g_1 + 3g_2 + g_3)(2g_2 + g_3) \\
&\quad -\frac{13}{4}g_3(g_1 + 3g_2 + g_3)^2 - 6g_2(g_1 + g_2)(2g_2 + g_3) \qquad (4.3)
\end{aligned}
$$

The Parisi-Sourlas fixed point is $g_1*^2 = -\frac{2\epsilon}{9}, g_2* = g_3* = 0$, and it is immediate to verify that in this case it is (marginally) stable. Therefore the mechanism which invalidates the epsilon-expansion for the RFIM is harmless for the problem of branched polymers.

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